# Possible Material Laws Including Length Dependences

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Traditional material laws of continuum mechanics of so-called simple materials always combine stresses $\sigma_{ij}$ of the dimension force per length$^2$ with strains $\varepsilon_{ij}$ being in principle dimensionless, in some cases with strain rates $\lambda_{ij}$ of dimension 1/time. A length will never be considered explicitly. The strain rates are determined as derivatives of displacements (lengths) with respect to space co-ordinates (lengths). Only if different kinematical variables, so e.g. angles, are taken into account, media like Cosserat-continua or media with long-range acting forces or with microstructures are included. In these media the width can play an important role. Their severe disadvantage is that they need much more strain measures, stresses, and material constants. Furthermore, a precise formulation of a nonlinear theory of this kind is only possible on the basis of a deep knowledge of tensor calculus.

## 1 Introduction, Basis of Linearly Elastic Material Laws

Also material laws of mechanics must be embedded in the thermodynamic framework. Especially, they must satisfy the second law of thermodynamics under arbitrary circumstances. It reads as Clausius-Dyhem inequality:

$$\underline{\underline{\sigma}} \cdot\cdot \underline{\underline{\lambda}} - \rho \frac{\partial f}{\partial \kappa_\alpha} \dot{\kappa}_\alpha - \frac{1}{\Theta} \underline{q} \cdot \nabla\Theta \geq 0 \, .$$

The quantities $\kappa_\alpha$ represent here mechanical quantities in the free energy $f$. Mostly, the in the original version only one inequality is satisfied by satisfying its three parts separately:

$$1)\ \underline{\underline{\sigma}} \cdot\cdot \underline{\underline{\lambda}} \geq \rho \frac{\partial f}{\partial \kappa_\alpha} \dot{\kappa}_\alpha \, , \quad 2)\ \Theta \geq 0 \, , \quad 3)\ \underline{q} \cdot \nabla\Theta \leq 0 \, .$$

The strain rates $\underline{\underline{\lambda}}$ read

$$\underline{\underline{\lambda}} = \tfrac{1}{2}[\nabla \otimes \underline{v} + (\nabla \otimes \underline{v})^T] \quad \text{or in co-ordinate notation} \quad \lambda_{ij} = \tfrac{1}{2}[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}] = \tfrac{1}{2}[v_{i|j} + v_{j|i}] \, .$$

The velocities $v_i$ are simultaneously strain rates of the displacements $u_i$, but the co-ordinates $x_i$, with respect to which is derived in $\nabla$ or in the covariant derivatives $_{|i}$, can change in time. This makes impossible – except for very small displacements and rotations – to adopt

$$\underline{\underline{\lambda}} = \dot{\underline{\underline{\varepsilon}}} \quad \text{with} \quad \varepsilon_{ij} = \tfrac{1}{2}[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}] \, .$$

But for such very small deformations the first inequality part yields

$$\sigma_{ij}\;\dot{\varepsilon}_{ji} \ge \rho \frac{\partial f}{\partial \varepsilon_{ji}} \dot{\varepsilon}_{ji} \qquad \text{with} \qquad \varepsilon_{ij} = \varepsilon_{ji} \,.$$

Hence, in the case of pure elasticity ( $\ge \longrightarrow =$ ) it leads to

$$\sigma_{ij} = \tfrac{1}{2}\rho\left(\frac{\partial f}{\partial \varepsilon_{ij}} + \frac{\partial f}{\partial \varepsilon_{ji}}\right),$$

with $\rho \approx \widetilde{\rho}$ and $\widetilde{\rho} f \longrightarrow \Psi$ (elastic potential) finally to

$$\sigma_{ij} = \tfrac{1}{2}\left(\frac{\partial \Psi}{\partial \varepsilon_{ij}} + \frac{\partial \Psi}{\partial \varepsilon_{ji}}\right).$$

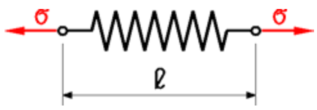


**Fig. 1:** A spring, possible symbol for elasticity

The behaviour symbolised by the Voigt-model of Fig. 1 with the potential

$$\Psi = \tfrac{1}{2}\varepsilon_{ij} C_{ijkl} \varepsilon_{kl} \qquad \text{where} \qquad C_{ijkl} = C_{jikl} = C_{klij}$$

holds is described by

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$$

including 21 numbers within the $C_{ijkl}$ in the case of full anisotropy. All the strains are dimensionless, and all the stresses have the same dimension. Hence, as Young has foreseen, no length is introduced anywhere. Only displacements grow proportional to the dimensions of a body. This holds even if other kinds of behaviour are considered.

## 2 Other Material Laws than Linearly Elastic Ones

Other types of material laws for simple materials may be only mentioned shortly here. Furthermore, only small deformations are to be considered. So, all the strains are additive. This is not true for large deformations. In the opposite, then it must be taken into account that the two deformation gradients $\underline{\underline{F}}_1$ and $\underline{\underline{F}}_2$ denoting two successive parts of any total motion are then connected in a multiplicative manner by

$$\underline{\underline{F}} = \underline{\underline{F}}_2 \cdot \underline{\underline{F}}_1 \,.$$

In the geometrically linear case it is possible to proceed according to the denotation of systems with spring or other elements in series or in parallel according to Figs. 2 ("viscosity") and 3 ("plasticity") or also in a much more complicated Voigt model for very sophisticated kinds of material behaviour. Nevertheless, no influence of any length can be recognised by such methods.

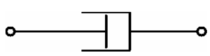

**Fig. 2:** A damper representing viscosity

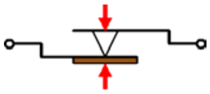

**Fig. 3:** A friction element representing plasticity

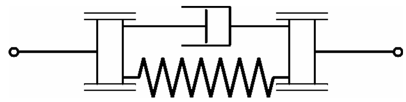

**Fig. 6:** Model for a medium with overstresses

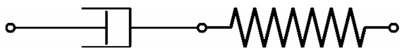

**Fig. 4:** A model for visco-elasticity

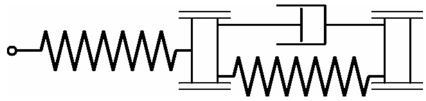

**Fig. 7:** Visco-elastic model

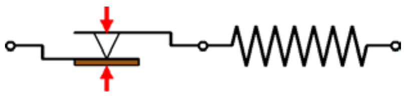

**Fig. 5:** A model for elasto-plasticity

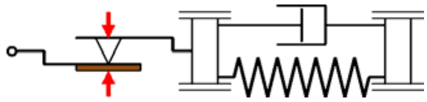

**Fig. 8:** Visco-plastic model with overstresses

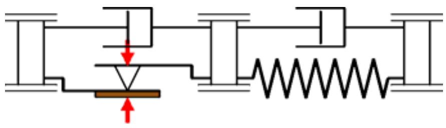

**Fig. 9:** Complicated visco-elastic-plastic model

In Figures 4 and 6, Voigt models for three cases of visco-elastic behaviour can be seen, Fig. 7 shows the typical elastic-plastic version, and Figs. 8 and 9 are due to two types of visco-elastic-plastic materials.

## 3 Strains and Stresses due to Large Deformations

A strict consideration of the necessities of continuum mechanics brings out easily that the strain $\underline{\underline{\varepsilon}}$ introduced above is in no sense satisfactory in the case of large deformations or even for large rigid body rotations. It has to be replaced by adequate (nonlinear) combinations of the deformation gradient $\underline{\underline{F}}$ :

$\underline{\underline{C}} = \underline{\underline{F}}^T \cdot \underline{\underline{F}} \rightarrow \underline{\underline{\gamma}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{1}})$ („Green-tensor") or $\underline{\underline{\varepsilon}} = \frac{1}{2}(\underline{\underline{1}} - \underline{\underline{F}}^{-T} \cdot \underline{\underline{F}}^{-1})$ („Almansi-tensor").

When applying the second law of thermodynamics, rates of these strains are needed, hopefully denoted in terms of the strain rates $\underline{\underline{\lambda}}$. This is easy in the case of $\underline{\underline{\gamma}}$ , since in connection with it everything is observed from a resting reference configuration (Lagrange-denotation). But when using $\underline{\underline{\varepsilon}}$ the variable actual configuration is considered (Euler-denotation). In this case, the observer has to move and (at least in average) to rotate like the material in the vicinity of the recognised material point. Then he sees exactly the „Zaremba-Jaumann rate“ $\overset{\oplus}{\underline{\underline{T}}}$ . So the following relations are produced:

$$\dot{\underline{\underline{\gamma}}} = \underline{\underline{F}}^T \cdot \underline{\underline{\lambda}} \cdot \underline{\underline{F}}, \qquad \overset{\oplus}{\underline{\underline{\varepsilon}}} = \underline{\underline{\lambda}} - \underline{\underline{\lambda}} \cdot \underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}} \cdot \underline{\underline{\lambda}}.$$

Also stresses are usually defined with respect to areas of the reference configuration. So the first and the second Piola-Kirchhoff-tensors are introduced, where the second one plays the most important role in numerical applications. It reads:

$$\underline{\underline{\widetilde{T}}} = \underline{\underline{F}}^{-1} \cdot \underline{\underline{\sigma}} \cdot \underline{\underline{F}}^{-T} \frac{dV}{d\widetilde{V}} \qquad \text{where} \qquad \frac{dV}{d\widetilde{V}} = \det(\underline{\underline{F}}).$$

Advantageously, it satisfies the following relation being fundamental for modern methods of analysis

$$\underline{\underline{\sigma}} \cdot\cdot \underline{\underline{\lambda}}\, dV = \underline{\underline{\widetilde{T}}} \cdot\cdot \dot{\underline{\underline{\gamma}}}\, d\widetilde{V}.$$

## 4 The Cosserat-continuum

Approximately, 100 years ago, the brothers Cosserat invented the director continuum, in which every material point has translatorical degrees of freedom like in the point continuum, but additionally also the rotatorical ones of a rigid body symbolised by a triad of directors. This may be illustrated for the case of plane motions mainly, but it was introduced by the Cosserats for the spatial case also from the very beginning. Also the formulae will be presented for the general case.

In Fig. 10, a point is now represented by a circle with a cross which can now rotate by an angle $\psi_z$ . In the spatial case of large rotations problems arise here, since there exists no additive vector of angles. Hence, it is difficult to denote the **three** rotatorical degrees of freedom. Here the use of quaternions might be advantageous.


**Fig. 10:** Rotation of a Cosserat-particle

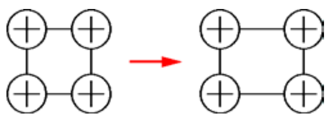

**Fig. 11:** Tension of Cosserat-media

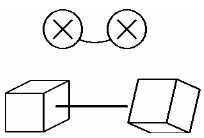

**Fig. 13:** Bending and torsion of Cosserat-media

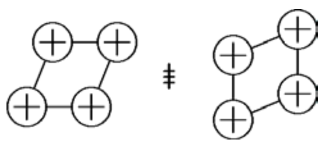

**Fig. 12:** Shear of Cosserat-media

Figure 11 shows, that the strains $\varepsilon_{xx}, \varepsilon_{yy}$, and $\varepsilon_{zz}$ exist also in the Cosserat-continuum. But, according to Fig. 12, it can differentiate between the shear deformations $\varepsilon_{xy}$ (left side of the figure) and $\varepsilon_{yx}$, since they do not differ by a rigid body rotation only like in a point-continuum. Now there are 9 strains already.

$$\varepsilon_{ij} = \frac{\partial u_i}{\partial x_j} + \varepsilon_{ijk}\psi_k \neq \varepsilon_{ji}.$$

Additionally this can not as simple be denoted in the geometrically nonlinear case. For some while there did not exist a consistent nonlinear Cosserat-theory. Here the habilitation thesis of Besdo may be mentioned.

Additionally to these 9 strains now also spatial variations of rotations (angles) must be recognised:

$$\kappa_{ij} = \frac{\partial \psi_i}{\partial x_j}.$$

Six of these 9 quantities denote (see upper part of Fig. 13) "bending" of the material (cf. beams), the other three ones – having equal indices – represent "torsion" (lower part of Fig. 13).

Totally, there are 18 strains of different dimensions, because the angles are dimensionless whereas the displacements are lengths and produce – like in the point-continuum – dimensionless strains by differentiations with respect to space co-ordinates. On the other hand, the strains denoting bending and torsion get the dimension length$^{-1}$.

Rotations of particles are forced by torques or couples applied to these particles. Hence, there are the usual forces acting distributed in inner surfaces which are now force-stresses. But additionally there are now also distributed couples named couple-stresses $\mu_{ij}$. Furthermore,

$$\sigma_{ij} = \sigma_{ji}$$

is no longer valid.

Hence, it is necessary now to distinguish strictly between the first index denoting the normal of the inner surface and the second one representing the direction of the force- or the moment vector. Please look at Fig. 14!

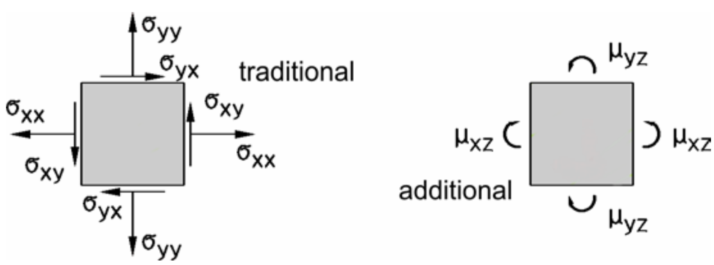


**Fig. 14:** Stresses and couple-stresses

Expressed in terms of velocities and strain rates, there are in a Cosserat-continuum

$$\lambda_{ij} = \frac{\partial v_i}{\partial x_j} + \varepsilon_{ijk}\omega_k \quad \text{and} \quad \upsilon_{ij} = \frac{\partial \omega_i}{\partial x_j}.$$

Together with the two types of stresses the produce the inner power being essential in the second law of thermodynamics. The purely mechanical part of this law now reads:

$$\sigma_{ij}\lambda_{ji} + \mu_{ij}\upsilon_{ij} \equiv \underline{\underline{\sigma}} \cdot\cdot \underline{\underline{\lambda}} + \underline{\underline{\mu}} \cdot\cdot \underline{\underline{\upsilon}} \geq \rho \frac{\partial f(...)}{\partial \kappa_{\alpha}} \dot{\kappa}_{\alpha} .$$

The strains $\varepsilon_{ij}$ and $\kappa_{ij}$ may now be the mechanical parameters $\kappa_{\alpha}$ of the free energy $f$. Especially in the geometrically nonlinear case it is not easy to express their derivatives by the so-called strain rates $\lambda_{ij}$ and $\upsilon_{ij}$.

In the linear-elastic case the following relations are produced:

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{lk} + D_{ijkl}\,\kappa_{lk} \quad \text{and} \quad \mu_{ij} = D_{ijkl}\,\varepsilon_{lk} + E_{ijkl}\,\kappa_{lk} .$$

The dimensions of the quantities appearing here are (Dim(...) = Dimension of ... ; F = force, L = length).

Dim($\sigma_{ij}$) = F/L$^2$, Dim($\mu_{ij}$) = FL/L$^2$= F/L , Dim($\varepsilon_{ij}$) = 1 , Dim($\kappa_{ij}$) = 1/L

and hence:

Dim($C_{ijkl}$)= F/ L$^2$, Dim($D_{ijkl}$) = F/L and Dim($E_{ijkl}$) = F.

Among these material constants there exists at least one characteristic length factor. Hence, dependencies from lengths can be recognised.

**A Problem** of applications of the Cosserat-theory is that, even in the case of an absolutely isotropic and linearly elastic material there are already **nine material numbers**, in a fully anisotropic one not less than 171.

Also viewing the balance equations are worth recognising: Like for point-continua the first equation is

$$\frac{\partial \sigma_{ij}}{\partial x_i} + f_j = \rho a_j .$$

In the Point-continuum the additional relation is $\sigma_{ij} = \sigma_{ji}$. But now a complete balance of couples has to be added looking very similar to the balance of forces

$$\frac{\partial \mu_{ij}}{\partial x_i} + \varepsilon_{jkl}\sigma_{kl} + m_j = \rho \frac{d}{dt}(\vartheta_{jk}\omega_k) .$$

Here the $m_j$ represent couples acting from outside the body per unit volume (e.g. at magnets), and the $\vartheta_{jk}$ denote mass moments per unit mass of the rigid particle (usually ignored). In the point-continuum there are neither couple-stresses nor couples acting from outside and also no mass-moments, hence, only the second term remains. It forces the relation usually called „Boltzmann-axiom". This points out that in a Cosserat-continuum it is much easier than in the classical "simple" one to understand that this fact often recognised as an axiom is nothing but the balance of torques produced by acting forces which had led to this second term.

A meaningful generalisation of the Cosserat-continuum is the general „director-continuum", for which the directors may not be rigid and rigidly connected. But this shall be mentioned only here.

## 5 Materials with Inner Long-Distance Forces

Inner forces do not act only between neighbouring atoms, but also over longer distances. But then they decrease strongly with the distances. In the theory of point-continua but also in Cosserat-theories it is a basic assumption that only direct neighbouring particles influence one-another. This is the reason that only in inner surfaces stresses appear and that they depend on relative motions of neighbouring particles only, hence only on the strains $\varepsilon_{ij}$ and on their rates. So it is astonishing how well this approach works.

Every elastic potential and every yield criterion expressed in term of a strain space, hence every scalar function appearing in any such material description depends on these $\varepsilon_{ij}$ and perhaps on the $\lambda_{ij}$ only. There exists no better proof for this than good experience. So in the middle of the last century people had the idea that (e.g.) the elastic potential could also depend on higher order spatial derivatives of the displacements. So they used new strains additional to the (dominating) old strains $\varepsilon_{ij}$. They will be denoted also by the letter $\kappa$ (now with three indices). But they have no correspondence to the appropriate quantities of the Cosserat-continuum:

In the geometrically linear case there also here the 6 independent quantities of the point-continuum:

$$\varepsilon_{ij} = \tfrac{1}{2}[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}] = \varepsilon_{ji}\,.$$

Additionally, there are 27 new quantities

$$\kappa_{ijk} = \frac{\partial^2 u_i}{\partial x_j \partial x_j}\,.$$

All of them may be significant. When writing down the balance of power new stresses $\mu_{ijk}$ being appropriate to the strains $\kappa_{ijk}$ must be introduced. In the geometrically linear case (small deformations) the mechanical part of the second law of thermodynamics then reads:

$$\sigma_{ij}\,\dot{\varepsilon}_{ji} + \mu_{ijk}\,\dot{\kappa}_{kji} \geq \rho \frac{\partial f}{\partial \kappa_\alpha}\dot{\kappa}_\alpha\,.$$

The $\kappa_\alpha$ can be the strains $\varepsilon_{ij}$ and $\kappa_{ijk}$. Then the material law for a linearly elastic medium is:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} + D_{ijklm}\kappa_{mlk} \quad \text{and} \quad \mu_{klm} = D_{klmij}\varepsilon_{ji} + E_{klmpqr}\kappa_{rqp}$$

This law needs much more than the nine numbers of the Cosserat-continuum in the isotropic case. The dimensions for the numbers $C_{ijkl}, D_{ijklm}, E_{klmpqr}$ are identical with those of the $C_{ijkl}$ to $E_{ijkl}$ of the Cosserat-continuum, hence, also this kind of continua can recognise length dependences (at least of appropriate types).

The determination of so many material numbers is a problem also in the linearly elastic case, the more it is one in case of other kinds of behaviour. Therefore, people have thought about simplifications. Like in the well-known joke where a drunken person searches for his keys which he has lost anywhere under a lantern since it is so nicely bright there, people – very often in the USA – try to use the old strains $\varepsilon_{ij}$ and additionally, as "representing" the action of non-neighbours with decreasing influence, the quantities $\Delta\varepsilon_{ij}$, where $\Delta$ is here the Laplace-operator, which is defined as

$$\Delta \ldots = \nabla \cdot \nabla \ldots$$

and can be translated relatively simply into every co-ordinate system. In kartesian x,y,z-co-ordinaten it produces the following six very special combinations of possible 81 ( ! ) strains which are expressed by third derivatives of displacements where everyone of them can be significant:

$$\kappa_{ij} = \varepsilon_{ij|kk} = \frac{\partial^2 \varepsilon_{ij}}{\partial x^2} + \frac{\partial^2 \varepsilon_{ij}}{\partial y^2} + \frac{\partial^2 \varepsilon_{ij}}{\partial z^2} .$$

This election is not really justified by physical reasons. But "very nice" results are created especially if these second derivatives – according to equations of the bending theory of second order – produce periodical solutions. This would – perhaps – also be yielded by a consequent application of a theory with the full set of first, second, and third derivatives and with appropriate reasonable material parameters and functions. Then there would be a chance to understand facts of the behaviour. As it is done it looks like an artefact not explaining anything.

In the view of mathematics this proceeding is similar to that of somebody who neglects all the even potencies in a Taylor-expansion without any necessity and uses only the terms with $x$, $x^3$, $x^5$.

Nevertheless, the stress-variables according to the $\Delta\varepsilon_{ij}$ cannot be interpreted really.

## 6 Conclusions

Normal continua which are continua of points do not present any chance to deal with length dependences. If effects like these have to be recognised it is necessary to use the "classical" Cosserat-continuum or more general director-continua. Also media with inner long-distance-forces represented by higher derivatives of displacements can be applied in this connection. But it is very doubtful to apply the often proposed theory with the Laplace-derivatives of usual strains. It produces – perhaps – nice (coloured) pictures, but it has no correct fundaments.